ON THE SEMIGROUP STRUCTURE OF CONTINUA(1)

BY

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Let S be a compact connected Hausdorff space. Suppose, moreover, that S is a topological semigroup. A continuum X is said to be aposyndetic, (Jones [8]), at a point x with respect to a point y if there is a subcontinuum M and an open set O such that $X-y\supset M\supset O\supset x$. It is well known, [8], that the set $T(p)$ is a continuum, where $T(p)$ denotes the set of points x such that X is not aposyndetic at x with respect to p.

Our first section will be devoted to a study of the sets $T(p)$ in S. A number of nonaposyndetic analogues of Faucett's results, [6], will be developed.

The results on the sets $T(p)$ will be applied to continua, irreducible between two points, with $S^2=S$. It will be shown that if S has a zero then S is an arc. This includes a result of [15]. The case in which S does not have a zero will be studied.

The results on the sets $T(p)$ and on irreducible continua will be applied to hereditarily unicoherent continua. It will follow, as a corollary, that if S is one dimensional with unit and zero then it is arcwise connected.

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We now list some of the standard terms used in the study of topological semigroups. A left (right) ideal is a nonvoid subset I such that $SI(IS)$ is a subset of I. The minimal ideal if it exists is denoted by K. It is known that if S is compact then K exists and is a retract of S. A subsemigroup A is a subset such that A^2 is contained in A. By a clan we mean a compact connected semigroup with unit. If e is an element such that $e^2=e$ then e is called an idempotent. The set E of all idempotents is closed. If A is a subset then $J(A)=A+SA+AS+SAS$. The set J_p is defined as the set of x such that $J(p)=J(x)$. If e is an idempotent then $H(e)$ denotes the maximal subgroup of e.

Again, we assume S to be compact and connected.

1. Definition 1. The set $T(p)$ is said to be symmetric if for any x in $T(p)$ it is true that p is a point of $T(x)$.

The first part of the following theorem was proved in [13] under the assumption S was a clan. For any set M, the symbol M^* will denote the closure of M.

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THEOREM 1.1. *If $S=ES+SE$ and $T(p)$ meets an ideal I then p is a point of I. If $T(p)$ is symmetric then $T(p)$ is contained in I.*

Proof. Suppose on the contrary that p is not in I. Let x be a point in the common part of I and $T(p)$. Let D be an open set about $J(x)$ such that D^* does not contain p. If J denotes the sum of the ideals of S contained in D then J is open by [16] and connected since $S=ES+SE$. But J^* is a continuum containing x within J, an open set. This is a contradiction.

The second statement follows from the first.

THEOREM 1.2. *If $S=ES$ and $T(p)$ meets the left ideal L then p is a point of $L+K$. If $T(p)$ is symmetric then $T(p)$ is contained in $L+K$.*

Proof. Similar to the above.

THEOREM 1.3. *If $S-T(p)=A+B$ mutually separate, $S=ES+SE$ and K is a subset of A then $J(p)$ is contained in A^*. If $T(p)$ is symmetric then $J(T(p))$ is contained in A^*.*

Proof. If J denotes the sum of the ideals contained in A then J is open (16). Since J^* is an ideal it meets $T(p)$. By Theorem 1.1 p is in J^* and the theorem follows.

It is easy to see that the restriction that $T(p)$ be symmetric for the second part of the above theorem is necessary.

DEFINITION 1.2. An ideal is said to be prime if its complement is a semigroup.

THEOREM 1.4. *Suppose $T(p)$ is symmetric, $S=ESE$, and that $S-T(p)$ $=A+B$, mutually separate. If A is a prime ideal then $T(p)$ is a group.*

Proof. If x and y are two points of $T(p)$ then xy is in $T(p)+B$ since A is prime. Now A^* contains $T(p)$ by Theorem 1.1 so that xy is in $T(p)$. Thus $T(p)$ is a compact semigroup and, as such, contains an idempotent e. Using Theorem 1.2 we see that eS and Se contain $T(p)$ so that e is a unit for $T(p)$. Furthermore e is the only idempotent in $T(p)$, for if f were another, one easily sees that $e=ef=f$, a contradiction. It now follows by [14] that $T(p)$ is a group.

The following notion will prove quite useful.

DEFINITION 1.3. The set C is said to weakly cut the set A from the set B if C meets every continuum which meets both A and B. If C is a point it is said to be a weak cut point.

THEOREM 1.5. *Suppose $T(p)$ is symmetric and weakly cuts the ideal A from the set B. If $S=ESE$ and $(T(p))^2$ meets $T(p)$ then $(T(p)+B)^2$ does not meet A.*

Proof. Suppose on the contrary that the points x and y are in $T(p)+B$ and that xy is in A. Since Sx meets both A and B it meets, and hence contains, $T(p)$. Likewise yS contains $T(p)$. Let c and d be points of $T(p)$ such that cd is in $T(p)$. Now $c=sx$ and $d=yt$ so that $cd=(sx)(yt)=s(xy)t$ is a point of A which is a contradiction.

The following is now immediate.

THEOREM 1.5. *Suppose $S=ESE$, the set $T(p)$ is symmetric, $S-T(p)=A+B$, mutually separate, and A is an ideal. If $(T(p))^2$ meets $T(p)$ then A is prime.*

DEFINITION 1.4. An ideal I is semi-prime if x^2 is not in I unless x is a point of I.

THEOREM 1.7. *Suppose $S=ESE$ and that $S-T(p)=A+B$, mutually separate. If A is a semi-prime ideal then A is prime.*

Proof. Since p^2 is not in A and since $J(p)$ is contained in A^* it follows that p^2 is a point of $T(p)$. Now suppose x and y are not in A but xy is. Since Sx meets $T(p)$ it follows that $p=sx$ for some s. Likewise $p=yt$. Now then, $p^2=(sx)(yt)=s(xy)t$ and hence is in A since A is an ideal. This is a contradiction.

Conditions in some of the previous theorems may be weakened or varied in accordance with the above.

THEOREM 1.8. *Suppose $S=ES+SE$ and p is in $S-K$. If $T(p)$ is symmetric it has vacuous interior.*

Proof. We note that $T(p)$ does not meet K. Suppose that $T(p)$ contains an open set O. If J denotes the sum of the ideals contained in $S-0^*$ then J^* meets the boundary of O. But then J^* contains O which is impossible.

In both the above theorem and the following it is easy to see that the condition of symmetry is essential.

THEOREM 1.9. *Suppose $S=ESE$ and p is in $S-K$. If $T(p)$ is symmetric and meets $H(e)$ then $T(p)$ is contained in $H(e)$.*

Proof. Since eS and Se both contain $H(e)$ and consequently $T(p)$ it is clear that $T(p)$ is contained in eSe. If x is any point of $T(p)$ then Sx contains $T(p)$ and hence contains $H(e)$. Likewise xS contains $H(e)$. We conclude that $T(p)$ is contained in $H(e)$.

The following shows an important connection between a weak cut point p and the set $T(p)$.

THEOREM 1.10. *Let X be a continuum and suppose p weakly cuts a from b. If $T(p)$ contains neither a nor b then $T(p)$ separates a from b.*

Proof. Suppose on the contrary that $T(p)$ does not separate a from b. For each point x of $X-T(p)$ there is a subcontinuum M and an open set O such that $S-p\supset M\supset O\supset x$. The collection of all such open sets forms a covering of $X-T(p)$ and so there is a simple chain $O_1, O_2, \cdots, O_n$ with O

containing a and O_n containing b. Since each O_i is contained in a continuum not containing p it follows that there is a continuum containing a and b but not p. This is a contradiction.

THEOREM 1.11. *Suppose $S=ES+SE$ and that A is the complement of a maximal prime ideal. If V is an open subset about A there is an open set U such that $A \subset U \subset V$ and $S-U$ an ideal and hence connected.*

Proof. For each point x of $S-A$ there is, as in Theorem 1.1, a subcontinuum M and an open set O such that $S-A \supset M \supset O \supset x$. Hence if V is open about A, the set $S-V$, since compact, is the sum of finitely many components. Since A does not separate S, an easy argument, similar to Theorem 1.10, shows the existence of the required set U.

DEFINITION. A continuum is said to be the essential sum of a collection G of continua if no element of G is contained in the sum of the others. A continuum is said to be n-indecomposable if it is the essential sum of n but not $n+1$ continua (Swingle, [20]).

We apply the set $T(p)$ to prove the following.

THEOREM 1.12. *If S is n-indecomposable and $S=ES+SE$ then $S=K$. Hence, if $n>1$, multiplication must be* (1) *$xy=x$ for all x, y or* (2) *$xy=y$ for all x, y.*

Proof. It is clear that if we form S/K, assuming K is proper, we see that there is an integer $r \leqq n$ such that S/K is r-indecomposable. Hence we may, without loss, assume S has a zero 0. Swingle, [20], has shown that S is the essential sum of n-indecomposable subcontinua $S_1, S_2, \cdots, S_n$. Let 0 be in S_1. Since $S=ES+SE$, it follows that for p, a point of S_1, the set $T(p)$ does not contain 0. It is shown in (2) that this is impossible. Now an easy argument shows that S cannot be the cartesian product of two nondegenerate continua. This together with the fact that $S=K$, shown above, and Corollary 1 of [16] implies the last statement of the theorem.

It is easy to see that if S is 1-indecomposable that any composant containing K is an ideal and that it contains E.

2. Throughout this section we shall assume S is a continuum irreducible between the points a and b. That is, no proper subcontinuum contains a and b.

We first examine the situation in which S has a zero and prove the following. By an arc from a to b, where a and b are points, we mean a continuum X containing a and b with the property that any point of X, other than a or b, separates a from b. In other words X is irreducibly connected between a and b.

THEOREM 2.1. *If $S^2=S$ and S has zero 0 then S is an arc. Either a or b is idempotent. If S has neither left nor right unit then both a and b are idempotent, $S-0=A+B$, mutually separate, and both A^* and B^* are abelian semigroups. If 0 does not separate S then S has a unit.*

Proof. Let us note first of all, that each set $T(p)$ is symmetric. This is quite easy to see since $S-T(p)$ has at most two components. Secondly, if S has neither left nor right unit, then it follows, from the irreducibility as in [15] that $S=eS+Sf$ and hence $S=eSe+fSf$ for e and f in E. Since, in this case, $a\neq 0\neq b$ it follows that $T(0)=\{0\}$ separates S. In any case we see that $S=ES+SE$. A straightforward argument using Theorems 1.1 and 1.3 shows that if $T(x)$ meets $T(y)$ then $T(x)$ is contained in, or contains $T(y)$. This argument is similar to the one we now use to show that if $T(x)$ contains $T(y)$ then $T(x)=T(y)$. We suppose then that $T(x)$ properly contains $T(y)$ and we may suppose, without loss of generality, that y is not in $T(a)+T(b)$. Then $S-T(y)=A+B$, mutually separate, with 0 an element of, say, A. By Theorem 1.2 the set $T(x)$ cannot meet B. Let p be a point of $T(x)$ which is in A but not $T(y)$. By symmetry, $T(p)$ does not contain y and it follows that $S-T(p)=C+D$ separate with 0 in C and y in D. Since x is in $T(p)$ we have a contradiction to Theorem 1.1 by means of Theorem 1.3. Hence the sets $T(p)$ are mutually exclusive, and an easy argument shows the collection, whose elements are the sets $T(p)$, to be upper semi-continuous. The hyperspace G is seen to be an arc by Theorem 1.10. Let e be an idempotent. We assert for any $T(x)$ in the interval, in G, from $T(0)$ to $T(e)$, that $T(x)=\{x\}$. We assume first that $T(x)$ contains no idempotent. Let $T(p)$ be the first element in the order from $T(x)$ to $T(e)$ such that $T(p)$ contains an idempotent f. It follows from [18] that in fSf an arc A may be started at f. If a is a point of A, by considering the locally connected continuum

$$\{A + aA + a^2A + \cdots + a^NA\}$$

for large enough N, and using the irreducibility of S, we see that $T(x)=\{x\}$. Now suppose that some $T(y)$, in the interval $T(0)$ to $T(e)$, is $T(g)$ for g in E. In the interval from $T(0)$ to $T(g)$ one cannot have sets $T(x)$, containing no idempotent, arbitrarily close to $T(g)$. Since in gSg there cannot be separating points arbitrarily close to g unless $H(g)=g$, there is a subinterval $T(h)$ to $T(g)$ each element of which is a set containing an idempotent. However, by Theorem 1.9 each set $T(p)$ contains at most one idempotent. Hence there is a cross section at e and by restricting the canonical mapping it follows that there is an arc from $T(h)$ to $T(g)$ and, finally, since there are not separating points close to g in gSg, unless $H(g)=g$, we conclude that $T(g)=\{g\}$. It is now clear that there is, in S, an arc from 0 to any idempotent e. Since $S=ES+SE$ it follows that S is an arc. The remaining statements are clear.

To simplify the discussion we use the notion of C-set as defined, for instance, in [21].

Definition 2.1. A subset M of a space X is called a C-set if any continuum meeting M and $X-M$ must contain M.

The boundary of a set A will always be denoted by $F(A)$. If there is a unique continuum irreducible from c to d it will be denoted by $[c, d]$.

LEMMA 2.1. *Suppose I is a closed subset of S, not separating S, such that S' (the space formed by shrinking I to a point) is an arc. If I has vacuous interior it is a C-set. If I is an ideal and S' has a unit* 1 *and if $F(S-I)$ is nondegenerate then $S-I$ is an abelian semigroup and $F(S-I)$ is an abelian group.*

Proof. The first conclusion is clear since each point of $S-I$ weakly cuts. To prove the second conclusion, let x and y be points of $S-I$ and suppose xy is in I. Now $x[y, 1]$ is a locally connected continuum containing x and meeting I. Since S' is an arc, it follows that $F(S-I)$ is degenerate. Since $S-I$ is abelian from [6] it follows that $(S-I)^*$ is abelian and its kernel is seen to be $F(S-I)$ which is then an abelian group.

In the remainder of this section, unless otherwise stated, we shall assume K is nondegenerate.

THEOREM 2.2. *Suppose K has vacuous interior. Then K is a group and if it does not separate S it is abelian and a C-set.*

Proof. The case in which K does not separate S follows from Lemma 2.1.

We assume K separates S so that $S-K=A+B$, mutually separate. If S has neither left nor right unit then from Theorem 2.1 we see that a and b are in E. If $F(A)$ is nondegenerate it is, by Lemma 2.1, a group. The same is true for $F(B)$. Clearly one or the other is nondegenerate. Since $K=F(A)+F(B)$ it follows that K is itself a group.

Let us suppose, now, that S has a left unit e which is in A. If $F(A)$ is degenerate then, letting $F(A)=k$, we see that the locally connected continuum $[k, e]b$ contains an arc from b to K and K is degenerate. Hence $F(A)$ is nondegenerate and an abelian group by Lemma 2.1. We may suppose $K-F(A)$ is nonvacuous. We note that $F(B)$ contains $K-F(A)$ and is nondegenerate so that by Lemma 2.1 $F(B)$ is a C-set in B^*. If x is any point of $F(B)$, by continuity of multiplication, we see that xS contains $F(B)$, as does Sx. Further, since $F(A)$, a group, meets xS it follows that xS contains K. It follows from [14] that K has a unit and consequently is a group.

THEOREM 2.3. *Suppose K has a nonvacuous interior. One of the following must hold:*

(i) *each element of K is a left zero.*

(ii) *each element of K is a right zero.*

(iii) *K is a group.*

Proof. It follows from [15] that our theorem will be proved if we can show that K is not the cartesian product of two nondegenerate continua. If K were such a product it would be aposyndetic [8]. By the irreducibility of S, any point of the interior of K would be a weak cut point of K. Such a point, by Theorem 1.10, is a separating point of K. Since K was a cartesian product this is impossible, and the theorem follows.

THEOREM 2.4. *If every element of K is a left zero and S has a left unit or has neither left nor right unit then K, and consequently S, is an arc.*

Proof. If S has left unit e and A is the component of e in $S-K$ then $F(A)$, if nondegenerate, is a group by Lemma 2.1. Hence we may assume $F(A) = \{k\}$ is degenerate. Taking $e=a$, and considering $[k, e]b$, the first part follows. In the second part we merely note that then $S=eSe+fSf$ and apply the first part.

If K is composed of left zeros and S has a right unit then any irreducible continuum may appear as K as in Example 2.3.

THEOREM 2.5. *If K is a group with nonvacuous interior then it is indecomposable.*

Proof. We assert that K is irreducible between two points.

We suppose first that K does not separate S and take, using Theorem 2.1, a as an idempotent and b a point of K. Now the boundary of $S-K$ is either degenerate or a group by Lemma 2.1 and, in either case, has vacuous interior in K. It follows that K is irreducible from b to any point of $F(S-K)$.

We now suppose that K separates S and write $S-K=A+B$, mutually separate. If S has neither left nor right unit then $F(A)$ is a group or degenerate and hence has no interior in K. The same is true for $F(B)$. It follows that K is irreducible from $F(A)$ to $F(B)$. If S has a left unit, say a, then $F(A)$ is nondegenerate and a group. (If not, we consider a translate by b of the arc from a to K.) Now $F(B)$ is a C-set in B^*. If $F(B)$ is degenerate it follows easily that K is irreducible from $F(B)$ to $F(A)$. If $F(B)$ is nondegenerate it follows that if x is in B and y is in A the product yx is in B, that is AB is contained in B. By continuity of multiplication it follows that if p is a point of $F(B)$ then $pF(A)$ contains $F(B)$. Hence $F(B)$ has no interior in K and it follows that K is irreducible from $F(A)$ to $F(B)$.

Since K is irreducible between two points and is homogeneous it follows from [3] that K is indecomposable.

We now list some examples.

EXAMPLE 2.1. It is shown in [15] that if G is any compact, connected, separable, abelian group then there is a clan, with kernel G, irreducible from G to the unit.

Throughout the following examples, I will denote the usual unit interval, $S\times T$ will denote the cartesian product with coordinatewise multiplication.

EXAMPLE 2.2. Let S be the clan of Example 2.1. Let C be a two point semigroup. Form $S\times C$ and shrink each set $g\times C$ to a point for g in G. That is, define $(s, c)R(s', c')$ if $s=c$ and $s'=c'$ or if $s=s'$ is in G. The usual methods show that S/R is a semigroup irreducible between two points. S may be described as a continuum group with two spirals winding upon it.

EXAMPLE 2.3. Let N be any continuum irreducible between two points c and d. For n and m in N define the product nm to be n. The semigroup

$(N\times\{0\})+(\{c\}\times I)+(\{d\}\times I)$ is irreducible between two points, has a right unit, and has N as kernel.

EXAMPLE 2.4. In [6] there is described a clan S irreducibly connected between two points k and e. Its kernel is nondegenerate and does not separate.

First form $S\times I$. We note that $(\{k\}\times I)+(S\times\{0\})+(\{e\}\times I)$ is a non-abelian clan, is an arc, and is separated by its kernel.

Secondly we note that the semigroup $(\{k\}\times I)+(S\times\{0\})$ is an arc, has a nondegenerate kernel which separates, and has neither left nor right unit.

EXAMPLE 2.5. Let S be, as in Example 2.1, irreducible from G to u. Forming $S\times I$ we see that $(\{e\}\times I)+(S\times\{0\})+(\{u\}\times I)$ is a clan which is irreducible.

Suppose G contains a subgroup H with the properties needed for Example 2.1. Form $S\times I$ and from the cylinder $H\times I$ construct T, irreducible as in Example 2.1. We see that $T+(S\times\{0\})$ is irreducible and has neither left nor right unit. Its kernel has vacuous interior but is not a C-set.

Concerning the above example, it is easy to see that if $S-K=A+B$, mutually separate S has a left unit e in A and $F(A)=K$, then $F(B)$ is either equal to K or is degenerate.

EXAMPLE 2.6. Let G be an indecomposable continuum which is a group. Forming $G\times I$ we note that $(G\times\{0\})+(\{e\}\times I)$, where e is the unit of G, is irreducible.

3. A continuum is said to be hereditarily unicoherent if the common part of two subcontinua, which intersect, is a continuum. If there is a unique continuum irreducible from the point c to the point d it will be denoted by $[c, d]$. It is obvious that in an hereditarily unicoherent continuum the subcontinua irreducible between two points are unique.

THEOREM 3.1. *If S is hereditarily unicoherent and has a unit* 1 *and a zero* 0 *then S is arcwise connected. Further, the arc* $[0, 1]$ *is an abelian semigroup.*

Proof. We shall show first that the continuum irreducible from 0 to 1 is a semigroup. Let x and y be points of $[0, 1]$ and suppose that xy is not an element of $[0, 1]$. We assert first that x is not an element of $[0, xy]\cap[0, 1]$. For if x were, the continuum $[0, x]y$ contains $[0, xy]$ which, in turn, contains $[0, x]$ since the irreducible continua are unique, and finally, $[0, x]y$ contains $[0, x]$ since xy is not in $[0, x]$. This is impossible by [14]. Hence we may suppose both x and y are not points of $[0, xy]\cap[0, 1]$. We now note that $\{[0, xy]+[0, 1]\}-\{[0, xy]\cap[0, 1]\}=A+B$, mutually separate, where xy is an element of A and x is an element of B. Since the continuum $x[y, 1]$ contains xy and x, and again by the uniqueness of the irreducible continua, we conclude that $x[y, 1]$ meets the continuum $x[y, 1]\cap[0, 1]$. Let z be a point in the common part of these continua. We note $z=xt$ for some t in $[y, 1]$.

We now assert that y is in tS. We suppose, on the contrary that y is not in tS. It is then clear by Theorem 1.2 that t is not in $T(y)$. We consider two cases: the first, when 1 is not in $T(y)$, the second, when 1 is in $T(y)$. In the first, we assert that either y weakly cuts t from 1 or that y weakly cuts 0 from t. If neither of these held, there would be a continuum containing 0 and 1 but not y which would be a contradiction to the uniqueness of the irreducible subcontinua. Now if y weakly cuts 0 from t it is immediate that y is in tS. Hence we may suppose that y weakly cuts t from 1. Since t is not in $T(y)$, and 1 is not in $T(y)$, it follows from Theorem 1.10 that $S-T(y)=A+B$, mutually separate, with t in A and 1 in B. Now $T(y)+B$ is a continuum containing y and 1 and consequently, $[y, 1]$. Since t was in $[y, 1]$, this is manifestly impossible. Hence we may suppose that 1 is in $T(y)$. Since $T(y)$ then contains $[y, 1]$ we conclude that t is in $T(y)$ which is a contradiction.

We now have shown that y is an element of tS so that $y=ts$ for some s. Now $xy=x(ts)=(xt)s$, so that $xy=zs$ with z in $[0, xy]\cap[0, 1]$. Finally, $[0, z]s$ contains $[0, zs]=[0, xy]$ which properly contains $[0, z]$ since xy is not in $[0, z]$. Since $[0, z]s$ properly contains $[0, z]$ we have a contradiction to [14]. Hence $[0, 1]$ is a semigroup and by Theorem 2.1 is an arc. By [6] we know that $[0, 1]$ is abelian. If c and d are two points of S, the locally connected continuum $c[0, 1]+d[0, 1]$, which is hereditarily unicoherent, obviously contains an arc $[c, d]$. It has been shown, in the proof of Theorem 5 of [13], for instance, that a one-dimensional clan with zero is hereditarily unicoherent.

COROLLARY. *Suppose $S^2=S$ and S has a zero. If S is hereditarily unicoherent it is arcwise connected. In particular, a one dimensional clan with zero is acrwise connected.*

Proof. The condition $S^2=S$ implies that $S=SES$ [16]. If s is any point of S then $s=xey$ for some e in E. Now the clan eSe, if one dimensional, is hereditarily unicoherent by [13]. Hence there is an arc $[0, e]$.By consideration of $x[0, e]y$ the theorem follows.

THEOREM 3.2. *Suppose S is arcwise connected and hereditarily unicoherent. If S has a zero 0 and x weakly cuts 0 from y then y is not in Sx.*

Proof. If $y=sx$, the continuum $s[0, x]$ properly contains $[0, y]$ in contradiction to [14].

THEOREM 3.3. *If S is arcwise connected, hereditarily unicoherent, and has a zero 0, then $p[0, q]=[0, pq]$ and $[0, p][0, q]=[0, pq]$.*

Proof. Clearly $p[0, q]$ contains $[0, pq]$. Suppose, for some x in $[0, q]$, that px is not in $[0, pq]$. Now $p[x, q]$ contains px and pq. Let z be the first point of $[px, 0]$ in the order from px to 0 which is a point of $[0, pq]$. We then have $z=py$ for some y in $[x, q]$ so that $x=ys$ for some s. But then, $px=p(ys)=(py)s=zs$ in contradiction to Theorem 3.2.

For the second conclusion we note first that $[0, p][0, q]$ contains $[0, pq]$ and we suppose that for some x in $[0, p]$ and y in $[0, q]$ the point xy is not in

$[0, pq]$. We note that $x[y, q]$ contains $[xy, xq]$ and that xq is an element of $[0, pq]$ by the first part of this theorem. If z is the first point of $[xy, xq]$ in $[0, pq]$ then $z = xt$ for t in $[y, q]$. Since $y = ts$, we see that $xy = zs$ in contradiction to Theorem 3.2.

THEOREM 3.4. *Let S be hereditarily unicoherent. If $S = ES + SE$ and has a zero 0 then x being in $T(p)$ implies p is not in $T(x)$.*

Proof. Obviously we may assume $x \neq 0 \neq p$. We assert that $[0, p] \cap T(p) = p$. Let y be any point in $[0, p] \cap T(p)$. It follows from Theorem 1.1 that $J(y)$ contains p which, by the usual argument, is impossible. Now suppose for some x in $T(p)$ that p is in $T(x)$. Now $T(p)$ contains $[p, x]$. Since $[0, p] \cap T(p) = p$ we conclude $[0, p] + [p, x] = [0, x]$ but by the first part, $T(x) \cap [0, x] = x$. We conclude that $x = p$.

DEFINITION 3.1. Let S be arcwise connected and hereditarily unicoherent. By an endpoint we mean a point which separates no arc.

The following theorem implies that a one dimensional continuum with unit and zero, which is a subset of the plane, is accessible at each of its nonzero endpoints from its single complementary domain.

A problem, which we leave unsolved, is when such a semigroup is accessible at its zero.

THEOREM 3.5. *Suppose S is hereditarily unicoherent and $S = ES + SE$. If p is an endpoint and p is not in K then S is semi-locally connected at p.*

Proof. Since we may form the Rees quotient S/K, we shall assume S has a zero 0. Let x be a point of $T(p)$. If x is in $[0, p] \cap T(p)$ then $x = p$. If x is not in $[0, p]$ then it follows that p separates the arc $[0, x]$. Hence $T(p) = p$. An easy argument shows that if V is an open set about p, there is an open set U containing p such that U is a subset of V, and $S - U$ is connected.

The notion of limiting set, $(A_\alpha \to A)$, as used in the following, is the usual one as described, for example, in [12].

THEOREM 3.6. *Suppose S is hereditarily unicoherent having a left unit e and a zero 0. If $\{A_\alpha\}$ is a collection of arcs each with 0 as an endpoint and $A_\alpha \to A$ then A is an arc.*

Proof. Let a_α be the nonzero endpoint of A_α and suppose $a_\alpha \to a$. By continuity, $[0, e]a_\alpha \to [0, e]a$. By Theorem 3.3, this implies $[0, a_\alpha] \to [0, a]$.

This theorem also follows from Theorem 3.4, but not in so direct a fashion.

THEOREM 3.7. *Suppose S is hereditarily unicoherent, has a zero 0, and a left unit. If one defines $x \leqq y$ if either x weakly cuts 0 from y or $x = 0$, then $\leqq$ is an order dense continuous partial order.*

Proof. It is clear that $\leqq$ is an order dense partial order. To show continuity, suppose $a \nleqq b$ and $b \nleqq a$. We assert that there exist open sets, U and

V about a and b such that $u \nleq v$ and $v \nleq u$ for u in U and v in V. Suppose on the contrary that there exist arbitrarily small U_α and V_α such that $u_\alpha \leqq v_\alpha$ for u_α in U_α and v_α in V_α. It then follows that $[0, v_\alpha] = [0, u_\alpha] + [u_\alpha, v_\alpha]$. Now the limiting set of $\{[0, v_\alpha]\}$, (or some subcollection), contains 0, a, and b, and, since it is an arc from the previous theorem, we conclude that 0 weakly cuts between a and b. Now S is locally connected at 0 so that we may find an open set D about 0 such that D^* is a continuum. An easy argument shows that for some β, a subarc of $[0, v_\beta]$ has its endpoints in D^* but is not a subset of D^*. Since this is impossible the proof is complete.

DEFINITION 3.2. By a maximal arc we mean one which is not a proper subset of any other arc.

It can be shown, using standard techniques, that in an arcwise connected hereditarily unicoherent continuum, any arc can be extended to a maximal arc. The situation is somewhat easier with a semigroup as the following shows.

THEOREM 3.8. *If S is hereditarily unicoherent with a left unit e and a zero* 0 *then S is arcwise connected and every arc is contained in a maximal arc.*

Proof. Arcwise connectedness follows from Theorem 3.3. Let $[0, a]$ be an arc. Let Q be the collection of all arcs of the form $[0, x_\alpha]$, α in A, where $[0, x_\alpha]$ contains $[0, a]$. Let T be a maximal tower in Q and let L be the closure of the union of the elements of T. Define $b_\alpha \leqq b_\beta$ if b_α weakly cuts 0 from b_β if and only if $[0, b_\alpha]$ is contained in $[0, b_\beta]$. Note that $(A, \leqq)$ is a directed set. Let b be a cluster point of $\{b_\beta\}$. By [14] and Theorem 3.3, we see that $L = (U[0, b_\beta])^* = (U[0, e]b_\beta)^* = [0, e]b = [0, b]$ and the theorem follows.

Concerning the existence of weak cut points we have the following.

THEOREM 3.9. *Suppose S is one dimensional and has a unit. If S has no weak cut point then S is a simple closed curve.*

Proof. If K is not proper form S/K. We know S/K is arcwise connected and hereditarily unicoherent. Let 0 be the zero of S/K. Since $S/K - 0$ and $S - K$ are homeomorphic it follows that S has a weak cutpoint. Hence we know that $S = K$ and consequently S is a topological group. Now S is certainly not indecomposable for every point of an indecomposable continuum is a weak cutpoint. Since S is a decomposable, one dimensional, compact, connected, topological group it is a simple closed curve.

THEOREM 3.10. *Suppose S is arcwise connected, hereditarily unicoherent and equal to $ES + SE$. If A is the complement of a maximal proper ideal M then every point of A is an endpoint and A is totally disconnected.*

Proof. We may assume S has a zero 0. Let a be in A and assume a is not an endpoint. Then $[0, a]$ is a proper subset of some $[0, t]$. Now $J(a) \cap [a, t] = a$. Since $M + J(a) = S$ we conclude $[a, t] - a$ is a subset of M. Since $J(t)$ is contained in M we see that a is in M which is impossible.

THEOREM 3.11. *Suppose S is arcwise connected, hereditarily unicoherent and has a zero. If the endpoints are idempotent and commute, one with another, then S is abelian.*

Proof. If r and s are any two points of S then r is in $[0, e]$ and s is in $[0, f]$ for some idempotents e and f. We note, from Theorem 3.3, that $[0, e][0, f] = [0, ef] = [0, fe] = [0, f][0, e]$ and that all of these contain rs and sr.

Now $rs = (re)(fs) = r(fe)s = (rf)(es)$. Both rf and es are points of $[0, ef]$ which is abelian by Theorem 3.1. Hence $(rf)(es) = e(sr)f$. We now assert that $e(sr)f = sr$. Ordering $[0, ef]$ from 0 to ef, we cannot have $e(sr)f < sr$, using Theorem 3.2, and were we to have $e(sr)f < sr$, then $f(e(sr)f)e = (fe)(sr)(fe) = sr$, again a contradiction to Theorem 3.2. Hence, S is abelian.

COROLLARY (FAUCETT). *If S is irreducibly connected between two idempotents which commute and has a zero then S is abelian.*

THEOREM 3.12. *Suppose S is hereditarily unicoherent with unit 1 and zero 0. If every endpoint is an element of $H(1)$ then $H(1)$ and $[0, 1]$ commute elementwise. Hence if $H(1)$ is abelian so is S.*

Proof. Using Theorems 3.1 and 3.3, the argument of [7] suffices.

We note that if S is hereditarily unicoherent and has a unit and a zero it has the fixed point property. This is, certainly in the metric case, well known [1]. It also follows from Theorem 3.7 and [22].

THEOREM 3.13. *Let S be one dimensional with unit 1. If S is not arcwise connected then K is a group.*

Proof. Since K is one dimensional it is not the cartesian product of two nondegenerate continua. Hence, [15], every element of K is a left (right) zero or K is a group. The continuum M, irreducible about $K + \{1\}$, is a semigroup by Theorem 3.1. Let $N = M - K$, and let $F(N)$ be the boundary of N in N^*. Since any point of N weakly cuts K from 1 it follows that $F(N)$ is a C-set in N^* and hence by Lemma 2.1, is a group if nondegenerate. Hence we may suppose $F(N) = \{k\}$ is degenerate. By considering translates of $[k, 1]$ the theorem follows.

By taking C as the Cantor set under "min" and G as the circle group in Example 2.2 the result is a totally nonaposyndetic clan which is one dimensional and has no separating point.

Let C be the Cantor set under "min" and S be a clan which is a triod with zero endpoint. Form $S \times C$ and shrink $\{0\} + C$ to a point. The resulting one dimensional clan with zero is not a subset of the plane.

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